The general functional form of the Biot-Savart law.

Author*: Ievgen Kandaurov*

Abstract: Although the law by which the field created by a conductor of arbitrary shape can be calculated has been known for a long time, there are not many specific solutions. These include solutions for the axis of a circular current or the axis of a solenoid. Less frequently, a general solution for a circular current can be found. In this work, based on the "Biot-Savart" law, general functions (functionals) are derived, the arguments of which are parametric functions and their derivatives describing the current curve in space. The result of the operation of these functionals are the specific functions Bx, By, Bz, which describe the components of the magnetic field vector at any given point in space.

Principle of Parallel Transport of the Field Source in Space:

Let us assume that for a fixed point $P(x, y, z)$, we know the functions describing the values of the magnetic field vector components at that point: Bx(x, y, z), By(x, y, z), Bz(x, y, z). We then parallelly move the field source along the x, y, z coordinates by values a, b, c, respectively. Since we have reduced the distance between the point of measurement and the source by the corresponding amounts, the new functions will take the form: Bx$(x - a, y - b, z - c)$, By$(x - a, y - b, z - c)$, Bz$(x - a, y - b, z - c)$

Finding the Field at Point $P(x, y, z)$ for a Section of Conductor on the x-axis:

We will write the Biot-Savart-Laplace law in scalar form, using the figure 1 (all calculations are carried out in the Gaussian system of units).

$$B = I \cdot \frac{\Delta a \cdot \sin(\phi)}{OP^2}$$

I – current flowing through the conductor

Δa – length of the conductor segment

OP – magnitude of the radius vector drawn from the current element to the field measurement point

ϕ – angle between the current direction vector and the radius vector $\overrightarrow{OP}$
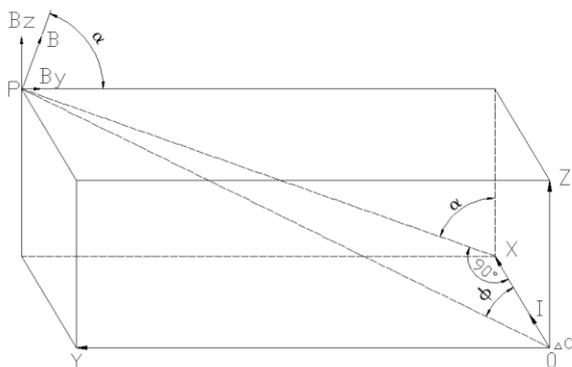
Before solving the problem, let us agree that the magnitudes of the vectors used in the problem will be expressed as functions of several variables, whose names will coincide with the names of the vectors. For example, the magnitude of the vector $\overrightarrow{OR}$ will be expressed by the function $OR(x, y, z)$ By the same principle, we will express the values of sines and cosines.

$$OP(x,y,z) = \sqrt{x^2 + y^2 + z^2} \qquad XP(y,z) = \sqrt{y^2 + z^2} \qquad Sin\phi(x,y,z) = \frac{XP(y,z)}{OP(x,y,z)} \qquad Sin\alpha(y,z) = \frac{y}{XP(y,z)}$$

$$B(x,y,z) = I \cdot \frac{Sin\phi(x,y,z)}{OP(x,y,z)^2} \cdot \Delta a \qquad Cos\alpha(y,z) = \frac{z}{XP(y,z)}$$

$$Bz(x,y,z) = B(x,y,z) \cdot Sin\alpha(y,z) = I \cdot \frac{y}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}} \cdot \Delta a$$

$$By(x,y,z) = -B(x,y,z) \cdot Cos\alpha(y,z) = -I \cdot \frac{z}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}} \cdot \Delta a$$

Similarly, we will solve the problem for the segments of the conductors lying along the y- and z-axes. Let us write the functions for these segments of the conductors.

The segment lying on the x-axis: $Xy(x,y,z)=By(x,y,z)$, $Xz(x,y,z)=Bz(x,y,z)$

$$Xy(x,y,z) = -I \cdot \frac{z}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}} \cdot \Delta a \qquad Xz(x,y,z) = I \cdot \frac{y}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}} \cdot \Delta a$$

The segment lying on the y-axis:

$$Yx(x,y,z) = I \cdot \frac{z}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}} \cdot \Delta b \qquad Yz(x,y,z) = -I \cdot \frac{x}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}} \cdot \Delta b$$

The segment lying on the z-axis:

$$Zx(x,y,z) = -I \cdot \frac{y}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}} \cdot \Delta c \qquad Zy(x,y,z) = I \cdot \frac{x}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}} \cdot \Delta c$$

Let the spatial current curve be described by the parametric functions $a(t), b(t), c(t)$.
a, b, c – are the coordinate axes coinciding with the x, y, z axes of the Cartesian coordinate system.
Δa, Δb, Δc – are the projections of a segment of the current curve onto the corresponding axes.
Using the principle of parallel translation and replacing Δa, Δb, Δc with the derivatives of these functions, we write the formulas for the current element:
$$dBx(a,b,c,db,dc,x,y,z) = Zx(x-a,y-b,z-c) \cdot dc + Yx(x-a,y-b,z-c) \cdot db$$
$$dBy(a,b,c,da,dc,x,y,z) = Zy(x-a,y-b,z-c) \cdot dc + Xy(x-a,y-b,z-c) \cdot da$$
$$dBz(a,b,c,da,db,x,y,z) = Xz(x-a,y-b,z-c) \cdot da + Yz(x-a,y-b,z-c) \cdot db$$

$$dBx(a,b,c,db,dc,x,y,z) = I \cdot \frac{dc \cdot (b-y) - db \cdot (c-z)}{\left[(a-x)^2 + (b-y)^2 + (c-z)^2 \right]^{\frac{3}{2}}}$$

$$dBy(a,b,c,da,dc,x,y,z) = I \cdot \frac{da \cdot (c-z) - dc \cdot (a-x)}{\left[(a-x)^2 + (b-y)^2 + (c-z)^2 \right]^{\frac{3}{2}}}$$

$$dBz(a,b,c,da,db,x,y,z) = I \cdot \frac{db \cdot (a-x) - da \cdot (b-y)}{\left[(a-x)^2 + (b-y)^2 + (c-z)^2 \right]^{\frac{3}{2}}}$$

a, b, c –parametric functions $a(t), b(t), c(t)$ describing the spatial curve.
da, db, dc – derivatives of the functions $a(t)`, b(t)`, c(t)`$
x,y,z – coordinates of the point where the field is measured.

Find the field at point $P(x, y, z)$, created by all elements of the current curve over the interval $t1 - t2$

$$Bx(a,b,c,db,dc,x,y,z) = I \cdot \int_{t1}^{t2} \frac{dc \cdot (b-y) - db \cdot (c-z)}{\left[(a-x)^2 + (b-y)^2 + (c-z)^2 \right]^{\frac{3}{2}}} dt$$

$$By(a,b,c,da,dc,x,y,z) = I \cdot \int_{t1}^{t2} \frac{da \cdot (c-z) - dc \cdot (a-x)}{\left[(a-x)^2 + (b-y)^2 + (c-z)^2 \right]^{\frac{3}{2}}} dt$$

$$Bz(a,b,c,da,db,x,y,z) = I \cdot \int_{t1}^{t2} \frac{db \cdot (a-x) - da \cdot (b-y)}{\left[(a-x)^2 + (b-y)^2 + (c-z)^2 \right]^{\frac{3}{2}}} dt$$

Example:

Let us find the field of a helical line. We make the following substitutions into the formulas obtained above:

$$t = \alpha \quad a = r \cdot \cos(\alpha) \quad b = r \cdot \sin(\alpha) \quad c = k \cdot r \cdot \alpha \quad da = -r \cdot \sin(\alpha) \quad db = r \cdot \cos(\alpha) \quad dc = k \cdot r$$

$$dBx(r\cdot\cos(\alpha), r\cdot\sin(\alpha), k\cdot r\cdot\alpha, r\cdot\cos(\alpha), k\cdot r, x, y, z) \rightarrow \frac{r\cdot\cos(\alpha)\cdot(z - \alpha\cdot k\cdot r) - k\cdot r\cdot(y - r\cdot\sin(\alpha))}{\left[(x - r\cdot\cos(\alpha))^2 + (y - r\cdot\sin(\alpha))^2 + (z - \alpha\cdot k\cdot r)^2\right]^{\frac{3}{2}}}$$

$$dBy(r\cdot\cos(\alpha), r\cdot\sin(\alpha), k\cdot r\cdot\alpha, -r\cdot\sin(\alpha), k\cdot r, x, y, z) \rightarrow \frac{r\cdot\sin(\alpha)\cdot(z - \alpha\cdot k\cdot r) + k\cdot r\cdot(x - r\cdot\cos(\alpha))}{\left[(x - r\cdot\cos(\alpha))^2 + (y - r\cdot\sin(\alpha))^2 + (z - \alpha\cdot k\cdot r)^2\right]^{\frac{3}{2}}}$$

$$dBz(r\cdot\cos(\alpha), r\cdot\sin(\alpha), k\cdot r\cdot\alpha, -r\cdot\sin(\alpha), r\cdot\cos(\alpha), x, y, z) \rightarrow -\frac{r\cdot\cos(\alpha)\cdot(x - r\cdot\cos(\alpha)) + r\cdot\sin(\alpha)\cdot(y - r\cdot\sin(\alpha))}{\left[(x - r\cdot\cos(\alpha))^2 + (y - r\cdot\sin(\alpha))^2 + (z - \alpha\cdot k\cdot r)^2\right]^{\frac{3}{2}}}$$

$$Bx(\alpha1, \alpha2, k, r, x, y, z) = I\cdot\int_{\alpha1}^{\alpha2} \frac{r\cdot\cos(\alpha)\cdot(z - \alpha\cdot k\cdot r) - k\cdot r\cdot(y - r\cdot\sin(\alpha))}{\left[(x - r\cdot\cos(\alpha))^2 + (y - r\cdot\sin(\alpha))^2 + (z - \alpha\cdot k\cdot r)^2\right]^{\frac{3}{2}}}\, d\alpha$$

$$By(\alpha1, \alpha2, k, r, x, y, z) = I\cdot\int_{\alpha1}^{\alpha2} \frac{r\cdot\sin(\alpha)\cdot(z - \alpha\cdot k\cdot r) + k\cdot r\cdot(x - r\cdot\cos(\alpha))}{\left[(x - r\cdot\cos(\alpha))^2 + (y - r\cdot\sin(\alpha))^2 + (z - \alpha\cdot k\cdot r)^2\right]^{\frac{3}{2}}}\, d\alpha$$

$$Bz(\alpha1, \alpha2, k, r, x, y, z) = -I\cdot\int_{\alpha1}^{\alpha2} \frac{r\cdot\cos(\alpha)\cdot(x - r\cdot\cos(\alpha)) + r\cdot\sin(\alpha)\cdot(y - r\cdot\sin(\alpha))}{\left[(x - r\cdot\cos(\alpha))^2 + (y - r\cdot\sin(\alpha))^2 + (z - \alpha\cdot k\cdot r)^2\right]^{\frac{3}{2}}}\, d\alpha$$

Conclusion: The obtained formulas provide a solution to an entire class of magnetostatics problems. The method of parallel translation of the source can be applied to solve more complex problems in which a current-carrying strip acts as the field source. This method is also applicable to solving problems in electrostatics.